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THE AVERAGED BERNOULLI EQUATION AND MACROSCOPIC EQUATIONS OF MOTION FOR THE POTENTIAL FLOW OF A TWO-PHASE DISPERSION

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Abstract—A description of a uniform two-phase dispersion in a potential flow is developed in terms of a macroscopic (averaged) potential and the average velocity of each phase. Using analogues to electromagnetic theory, the constitutive laws are expressed in terms of either an effective conductivity of the matrix or its polarization (average dipole moment). Bernoulli's equation for the fluid is expressed in averaged form. The results are applied to three example problems. The equations of motion for the "two-fluid model" are derived using the hypothesis that the averaged Bernoulli equation is differentiable. The continuous phase is incompressible but the dispersed phase may be compressible. The results are compatible with Geurst's equations which were derived using variational methods.

Key Words: two-phase, two-fluid, dispersion, potential, Bernoulli, volume-average, exertia, added mass, equations of motion

INTRODUCTION

A major research objective of the past two decades has been the derivation of macroscopic conservation laws for two-phase flow that could provide the starting point for the solution of general problems, playing a role equivalent to the Navier–Stokes equations in single-phase flow.

The usual approach has been to average the single-phase conservation laws (e.g. Delhaye 1968; Ishii 1975). While this leads to some very general forms of equations, it leaves unresolved certain terms in the equations that have to be specified by separate "closure hypotheses".

In a bold and innovative approach, Geurst (1985, 1986) was able to derive macroscopic equations of motion using variational methods. This bypassed the need for separate closure laws but left unanswered the question of how his results were related to the averaging approach. Moreover, strict conditions for validity were not established.

Wallis (1989a) derived specific solutions to a variety of situations using the methods of classical potential flow theory. In every case where comparisons could be made, his results agreed with Geurst's. Though no general equations of motion were derived, the macroscopic variables that were deduced were related to area and volume averages, incorporating, for example, the average stresses in the dispersed phase and the Reynolds stresses in the continuous phase.

In the present paper it is shown how potential flow theory can be used to derive an averaged "Bernoulli equation" for the incompressible continuous phase which is identical to one of Geurst's results. This can then be used, together with the overall momentum balance for the combined phases, to derive "equations of motion" which agree with Geurst's and provide independent evidence for their validity. The dispersed phase may be compressible.

The usefulness of developing a self-consistent theory of two-phase potential flow has been discussed elsewhere (Wallis 1989a). By developing specific expressions for constitutive equations and macroscopic properties such as average pressures, stress tensors, Reynolds stresses and so on for a well-defined ideal system, the interrelationships between these quantities may be more clearly examined and tested. Moreover, the results are useful in practice for analyzing the same sorts of "limiting cases" for which classical potential flow is valid; e.g. inertia-dominated flows with rapid acceleration, small amplitude oscillations and acoustic wave transmission. The inclusion of viscous effects, which are important in many practical situations, is a much more difficult problem in which empiricism will probably play a key role. However, some progress in this direction might be achieved by using potential theory as a starting point and incorporating viscous effects when they are small or confined to regions such as "boundary layers".

AVERAGED BERNOULLI EQUATION FOR THE FLUID

We begin with the "base case", described by Wallis (1989a, section 3.1), with fluid, occupying a fraction ϵ_1 of the total volume, flowing at an average velocity $\mathbf{w}$ past stationary particles. The net fluid flux is

$$\mathbf{j}_0 = \epsilon_1 \mathbf{w}. \qquad [1]$$

For purposes of conceptualization, the flow may be considered to be in a duct, sketched in figure 1, with uniform area A and length L, all macroscopic dimensions being much larger than the particle size. "End effects" and "edge effects" can be made as small as desired by using a sufficiently large volume.

Because of the linearity of Laplace's equation, the macroscopic potential gradient is related to the average fluid velocity, or alternatively to the fluid flux, by

$$-\nabla \Phi_0 = \frac{\phi_B - \phi_A}{L} \hat{\mathbf{n}} = \epsilon_1 \beta \mathbf{w} = \beta \mathbf{j}_0, \qquad [2]$$

where β is a function of the volume fraction of particles, ϵ_2, their shape and geometrical arrangement, and is a scalar quantity in an isotropic suspension. β could be thought of as the inverse of the "conductance" or "porosity" of the matrix. The subscript "0" denotes the "base case" where particles are at rest.

Though figure 1 is useful as an example of a simple operational definition of β, [2] is a fundamental relationship describing the properties of the matrix itself. This "constitutive equation" is directly analogous to the classical solutions of the problem of electrical conduction past a matrix of dispersed particles considered by, amongst others, Maxwell (1881), Rayleigh (1892) and Jeffrey (1973).

Without the particles, the fluid velocity everywhere would be $-\nabla \Phi_0$. Therefore, the effect of the particles is to create a superposed potential that could be ascribed to sources in the particles, causing the resultant streamlines to flow around the particles. A sketch of the resulting flow field when no external potential is applied and the particle "swims" along, carrying fluid with it by "drift", is shown for a special case in Wallis (1989a, figure 3.4).

Should the particles be "generated" by internal sources and sinks, no streamlines pass through the surface of each particle and there is therefore no net internal source. At a distance great compared with the particle size, which is the appropriate viewpoint for a "continuum" approach, each particle will appear as a dipole. The assembly resembles magnetic or electric grains polarized in an applied external field. The dipole strength per unit volume, $\mathbf{d}$, will depend on the particle geometry and particle concentration but, in an isotropic assembly, will be proportional to the applied fluid velocity or flux:

$$\mathbf{d} = -\delta \mathbf{j}_0. \qquad [3]$$

This dipole density produces a macroscopic "reactive" potential gradient:

$$\nabla \Phi_R = \mathbf{d} = -\delta \mathbf{j}_0. \qquad [4]$$

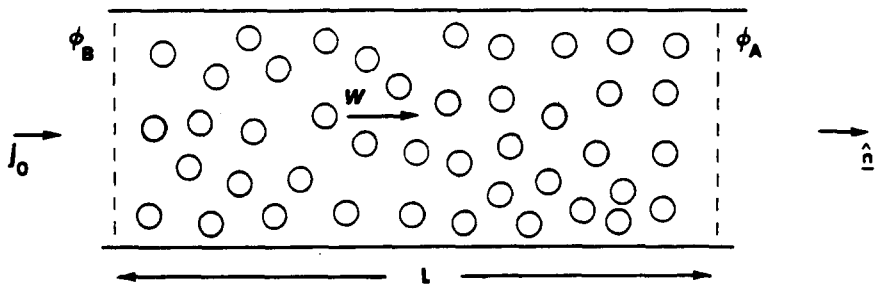

Figure 1. The "base case": flow past a stationary particle matrix.

Now, without the dipoles, the potential gradient is

$$\nabla\Phi_j = -\mathbf{j}_0 \qquad [5]$$

and, since no flux is added by inserting the dipoles (the emitted streamlines are all closed within the particles), superposition yields the average net potential gradient as

$$\nabla\Phi_0 = \nabla\Phi_R + \nabla\Phi_j = -(1+\delta)\mathbf{j}_0. \qquad [6]$$

Therefore the "inverse porosity", β, is related to the dipole density by

$$\beta = 1 + \delta. \qquad [7]$$

This derivation provides a method of computing β for simple arrays of spheres, e.g. by the method of images. Moreover, it establishes a key link to classical works of Lamb (1932) and Taylor (1928) in which the added mass for an object in a fluid is derived from the net dipole moment of its internal sources. [The "doublet" strength used by Lamb (1932) is $1/(4\pi)$ times the dipole strength used here which is more consistent with modern field theory.]

The equivalence of the dipole density approach and the resistivity approach may be illustrated by comparing the solution of Biesheuvel & Spoelstra (1989), for the problem of sudden acceleration of a dispersion of bubbles, with Jeffrey's (1973) solution for effective conductivity. In Biesheuvel & Spoelstra (1989) $\mathbf{v} - \mathbf{u}$ is equivalent to the present $-\mathbf{j}_0$ and their dipole density is

$$\delta = \frac{3}{2}\epsilon_2 + \frac{3}{2}\epsilon_2^2\left(1 + \sum\right), \qquad [8]$$

where

$$\sum = \sum_6^\infty \frac{B_p - 3A_p}{(p-3)2^{p-3}} \qquad [9]$$

is a summation defined in the referenced papers. The first term in [8] is the contribution from individual spheres, while the second term describes "interactions" or reflections. Using this value of δ in [7], the effective conductivity is, to second order,

$$\frac{1}{\beta} = 1 - \frac{3}{2}\epsilon_2 + \frac{3}{2}\left(\frac{1}{2} - \sum\right)\epsilon_2^2, \qquad [10]$$

which is Jeffrey's (1973) result.

In order to derive the properties of the "fluctuations" in the fluid flow induced by the presence of the particles, the microscopic potential everywhere in the fluid can now be expressed as the combination of an average and a varying part:

$$\phi = \Phi_0 + \phi_0. \qquad [11]$$

If we denote by $\langle\ \rangle$ the operation of averaging over a volume large compared with the particles, we have, from [11]:

$$\langle\phi\rangle = \Phi_0 + \langle\phi_0\rangle. \qquad [12]$$

Since Φ_0 is the average potential, it follows that

$$\langle\phi_0\rangle = 0. \qquad [13]$$

The local fluid velocity is

$$\mathbf{u}_0 = -\nabla\phi = -\nabla\Phi_0 - \nabla\phi_0. \qquad [14]$$

Averaging this equation and using [2] we get

$$\langle\mathbf{u}_0\rangle = \mathbf{w} = \epsilon_1\beta\mathbf{w} - \langle\nabla\phi_0\rangle. \qquad [15]$$

Therefore

$$\langle\nabla\phi_0\rangle = \mathbf{w}(\epsilon_1\beta - 1) = E\mathbf{w}, \qquad [16]$$

where

$$E = (\epsilon_1 \beta - 1) \tag{17}$$

is the "exertia" defined by Wallis (1989a, below equation [3.16]). Geurst's "m" is equal to $\epsilon_1 E$.

For all the fluid between the equipotentials ϕ_A and ϕ_B in figure 1,

$$\int u_0^2 \, dV = -\int \phi \mathbf{u}_0 \cdot d\mathbf{s} = (\phi_B - \phi_A) A \epsilon_1 w. \tag{18}$$

Therefore, using [2] and [1],

$$\langle u_0^2 \rangle = \frac{\int u_0^2 \, dV}{\epsilon_1 AL} = \epsilon_1 \beta w^2. \tag{19}$$

When [14] is substituted into the l.h.s. of [19] and [15] is used to evaluate $\langle \mathbf{u} \rangle$, there results

$$\langle (\nabla \phi_0)^2 \rangle = \epsilon_1 \beta (\epsilon_1 \beta - 1) w^2 = E(E+1) w^2. \tag{20}$$

Equations [13], [16] and [20] are useful for obtaining "macroscopic" laws by averaging equations representing conservation laws at the microscopic level.

ϕ_0 is the fluctuating part of the potential which emanates from the particles as a reaction to the relative velocity, $\mathbf{w}$. When the particles move, this potential field moves with them, just as the dipolar field associated with a single sphere moving in an infinite fluid moves as if attached to the particle. In principle it ought to be possible to write

$$\phi_0 = \mathbf{f} \cdot \mathbf{w}, \tag{21}$$

where $\mathbf{f}$ is an isotropic function that is clearly interpretable for the dipolar field of a single sphere [and also for a sphere moving in a flow with a uniform rate of strain (Voinov 1973)] but requires some probabilistic interpretation if it is to describe a random array of spheres, for example.

Now, let the entire pattern in the base case move with a velocity $\mathbf{v}_2$ which is the velocity of every particle. This is equivalent to superposing a potential ϕ_2 such that

$$-\nabla \phi_2 = \mathbf{v}_2. \tag{22}$$

The average velocity of phase 1, the fluid, is now

$$\mathbf{v}_1 = \langle u \rangle = -\langle \nabla (\phi_2 + \Phi_0 + \phi_0 \rangle. \tag{23}$$

The new macroscopic potential is

$$\Phi = \phi_2 + \Phi_0 \tag{24}$$

and [16] may be used in [23] to show that

$$-\nabla \Phi = \mathbf{v}_1 + E\mathbf{w}. \tag{25}$$

The "reaction" from the particles, as a result of the relative velocity,

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2 \tag{26}$$

must be overcome by an additional potential gradient, $E\mathbf{w}$, in order to move the fluid.

In the particular case where there is no macroscopic potential gradient, [25] and [26] may be combined to give

$$\mathbf{v}_1 = -E\mathbf{w} \tag{27}$$

and

$$\mathbf{v}_2 = -\alpha_1 \beta \mathbf{w}. \tag{28}$$

The ratio of the fluid flux to the particle flux is then

$$C = \frac{-E\mathbf{w}\epsilon_1}{-\epsilon_1 \beta \mathbf{w} \epsilon_2} = \frac{E}{\epsilon_2 \beta} = \frac{\epsilon_1 - \dfrac{1}{\beta}}{\epsilon_2} \tag{29}$$

which is perhaps the most straightforward definition of the "added mass coefficient" (Wallis 1989a, equation [3.34]) as it represents the volume of fluid carried along by unit volume of particles when there is no applied macroscopic potential gradient.

We now consider a more general motion in which $\mathbf{v}_1$ and $\mathbf{v}_2$ can vary with time while the particle matrix remains large and uniform with a fixed structure.

The microscopic Bernoulli equation for anywhere in the flow is, if the fluid is incompressible,

$$-\frac{\partial \phi}{\partial t}+\frac{u^2}{2}+\frac{p}{\rho}+gz=C[t]. \qquad [30]$$

Expressing ϕ and $\mathbf{u}$ as the values for the base case plus the contribution from the imposed motion with velocity $\mathbf{v}_2$ we get

$$-\frac{\partial}{\partial t}(\phi_2+\Phi_0+\phi_0)+\frac{(\mathbf{u}_0+\mathbf{v}_2)\cdot(\mathbf{u}_0+\mathbf{v}_2)}{2}+\frac{p}{\rho}+gz=C[t]. \qquad [31]$$

Now, $\partial\phi_0/\partial t$ is made up of two parts, one due to temporal changes in $\mathbf{w}$ and one due to the motion of the particles carrying their associated potential pattern at a velocity $\mathbf{v}_2$. Using [21] we have

$$\frac{\partial}{\partial t}(\phi_0)=\mathbf{f}\cdot\frac{\partial \mathbf{w}}{\partial t}-\mathbf{v}_2\cdot\nabla\phi_0. \qquad [32]$$

$\partial\mathbf{f}/\partial t$ is zero because at this stage there is no change in the structure of the particle matrix. There is also no term involving $\mathbf{v}_2\cdot\nabla\mathbf{w}$ because we are considering a locally uniform flow pattern. Substituting [32] in [31] and rearranging gives

$$-\frac{\partial}{\partial t}(\phi_2+\Phi_0)-\mathbf{f}\cdot\frac{\partial \mathbf{w}}{\partial t}+\frac{u_0^2}{2}+\frac{v_2^2}{2}+\mathbf{v}_2\cdot(\mathbf{u}_0+\nabla\phi_0)+\frac{p}{\rho}+gz=C[t]. \qquad [33]$$

The first term in parentheses in [33] is simply Φ from [24], and the second term in parentheses is $\epsilon_1\beta\mathbf{w}$, from [2] and [14]. Therefore, [33] may be simplified and averaged throughout the fluid to give

$$-\frac{\partial \Phi}{\partial t}+\frac{\epsilon_1\beta w^2}{2}+\frac{v_2^2}{2}+\mathbf{v}_2\cdot\epsilon_1\beta\mathbf{w}+\frac{p_1}{\rho}+gz=C[t], \qquad [34]$$

where [19] was used and $\langle\mathbf{f}\rangle$ set equal to zero in view of [10] and [21]. In [34], z is the elevation of the centroid of the fluid over which averaging has been performed.

Equation [34] is one form of the *averaged Bernoulli equation* for the fluid when the particles are moving uniformly. Whether it is valid, and to what order of accuracy, when there are gradients in the macroscopic properties may be debated, but it certainly applies between regions of a flow which are each in a uniform state, such as the inlet and outlet regions of a nozzle. The key is [30] which is valid *anywhere* in the fluid.

Equation [34] may be expressed in several equivalent forms. One variant may be obtained by substituting for $\mathbf{v}_2$ using [26],

$$-\frac{\partial \Phi}{\partial t}+\mathbf{v}_1\cdot(\mathbf{v}_1+E\mathbf{w})-\frac{v_1^2}{2}-E\frac{w^2}{2}+\frac{p_1}{\rho}+gz=C[t], \qquad [35]$$

which is in a form derivable (Wallis 1989b) from Geurst's (1985, 1986) variational approaches that have previously been found to be compatible with results obtainable from potential flow theory (Wallis 1991). It is notable, in view of [25], that the first two terms of [35] are the total derivative of $(-\Phi)$ in a coordinate system moving at the average fluid velocity.

An additional variant may be obtained by combining the terms involving E in [35], realizing that

$$\mathbf{v}_1\cdot\mathbf{w}-\frac{w^2}{2}=\left(\mathbf{v}_1-\frac{\mathbf{v}_1-\mathbf{v}_2}{2}\right)\cdot(\mathbf{v}_1-\mathbf{v}_2)=\frac{v_1^2-v_2^2}{2}. \qquad [36]$$

The result is

$$-\frac{\partial \Phi}{\partial t}+\frac{v_1^2}{2}+\frac{E}{2}(v_1^2-v_2^2)+\frac{p_1}{\rho}+gz=C[t]. \qquad [37]$$

Again, this differs from a one-dimensional version by a "correction" proportional to E. Part of the correction arose from averaging the u^2 term in [30]. The origin of the remainder was the convective term in $\partial\phi/\partial t$ due to the particle motion.

EXAMPLES

In order to illustrate and test the general results presented up to now, three simple examples of their application will be presented.

1. Motion of a porous sphere in an infinite fluid

The geometry of the flow to be considered is shown in figure 2. A porous sphere composed of a uniform matrix of particles occupying a fraction ϵ_2 of its volume moves at velocity $\mathbf{v}_2$ in the z-direction in an unbounded incompressible inviscid fluid that is at rest far from the sphere.

The fluid velocity fields resemble those found in several analogous situations in electromagnetic theory. The field is dipolar outside the sphere and uniform, with average velocity $\mathbf{v}_1$, parallel to $\mathbf{v}_2$, inside the sphere. There is therefore a uniform average dipole density $\mathbf{d}$, described by [3], inside the sphere, parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ and given explicitly, using [2] and [1], by

$$\mathbf{d} = -(\beta - 1)\epsilon_1 \mathbf{w} \qquad [38]$$

The external field is that from a concentrated dipole at the center of the sphere equal in strength to the net dipole moment of the sphere, therefore the velocity potential is

$$\Phi = \mathbf{d} \cdot \mathbf{r} \frac{a^3}{3r^3} \qquad [39]$$

and the velocity components are, in the spherical coordinates indicated,

$$v_r = \frac{2}{3}\frac{a^3}{r^3} d \cos\theta, \quad v_\theta = \frac{1}{3}\frac{a^3}{r^3} d \sin\theta. \qquad [40]$$

On the surface of the sphere,

$$\Phi = \frac{a}{3} d \cos\theta = \frac{d}{3} z \qquad [41]$$

and

$$v_r = \tfrac{2}{3} d \cos\theta, \quad v_\theta = \tfrac{1}{3} d \sin\theta. \qquad [42]$$

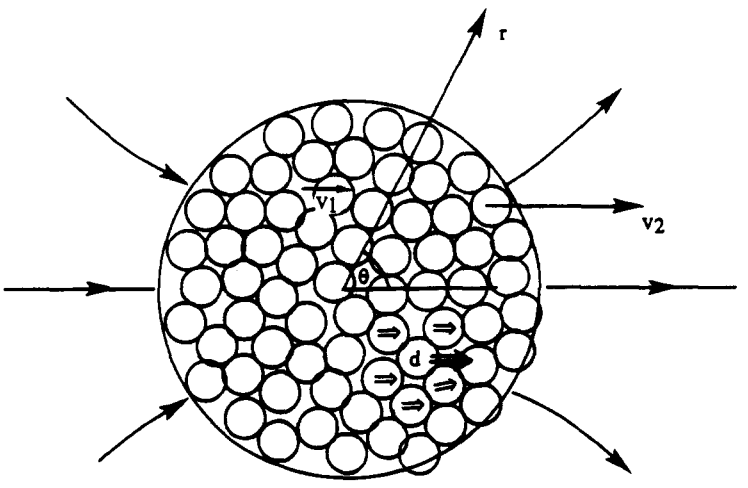

Figure 2. Motion of a uniform porous sphere in an infinite fluid; $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{d}$ are the average fluid velocity, solid velocity and average dipole density inside the sphere.

The net volumetric flux inside the sphere is

$$\mathbf{j} = \epsilon_1 \mathbf{v}_1 + \epsilon_2 \mathbf{v}_2 = \mathbf{v}_2 + \epsilon_1 \mathbf{w}. \qquad [43]$$

The condition at the surface of the sphere is that the radial components of volumetric flux should be continuous. Using [42] and [43], resolved in this direction, we have

$$\tfrac{2}{3} d \cos\theta = (v_2 + \epsilon_1 w)\cos\theta. \qquad [44]$$

Using [38] in [44] we may now solve for v_2 in terms of d to obtain

$$v_2 = \frac{d}{3}\frac{2\beta + 1}{\beta - 1}. \qquad [45]$$

For a Maxwellian dispersion ((Maxwell 1881; Wallis 1989a), equivalent to neglecting Σ in [8]),

$$\beta = \frac{\frac{\epsilon_2}{2} + 1}{\epsilon_1} \qquad [46]$$

and [45] reduces to

$$v_2 = \frac{2}{3}\frac{d}{\epsilon_2}. \qquad [47]$$

It may now be checked that [2] is indeed valid in the sphere. Using [6] and [26] in [2], we need to show that

$$-\frac{d}{3} = \epsilon_1 \beta w + v_2. \qquad [48]$$

The r.h.s. of [48] may be evaluated, using [38] and [45], as

$$-\frac{d\beta}{\beta - 1} + \frac{d}{3}\frac{2\beta + 1}{\beta - 1} = -\frac{d}{3}, \qquad [49]$$

which confirms [48] and shows that the "resistivity" approach is consistent with the "polarization" approach which was used to derive [45].

The average fluid velocity in the sphere follows from [26], [38] and [45] as

$$v_1 = d\frac{2\epsilon_1\beta + \epsilon_1 - 3}{3\epsilon_1(\beta - 1)} \qquad [50]$$

or, in terms of v_2, using [45],

$$v_1 = v_2\left[1 - \frac{3}{\epsilon_1(2\beta + 1)}\right], \qquad [51]$$

which reduces, for a Maxwellian dispersion, to $v_1 = 0$(!).

The "added mass" of the sphere is its net dipole moment, divided by its velocity, minus its displaced volume (Lamb 1932), i.e.

$$\frac{4}{3}\pi a^3\left(3\frac{\beta - 1}{2\beta + 1} - \epsilon_2\right). \qquad [52]$$

The factor in parentheses in [52] could be called the "added mass coefficient". For a Maxwellian dispersion it reduces to $\epsilon_2/2$, and is consistent with the limiting cases of a solid sphere or a sphere composed of a very dilute suspension. Indeed, for this particular case the added mass is merely one-half of the mass displaced for all particle concentrations.

Equation [52] may be confirmed by evaluating the total kinetic energy of the fluid. Outside the sphere, in view of [45], it is the same as for fluid surrounding a solid sphere with radius a, volume $V = 4\pi a^3/3$ and speed $2v_2(\beta - 1)/(2\beta + 1)$. This contributes kinetic energy equal to

$$K_{\text{out}} = V\frac{\rho_1}{2}v_2^2 2\left(\frac{\beta - 1}{2\beta + 1}\right)^2. \qquad [53]$$

Inside the sphere the fluid kinetic energy is (Wallis 1989a):

$$K_{in} = V\frac{\rho_1}{2}[\epsilon_1 v_1^2 + \epsilon_1(\epsilon_1\beta - 1)w^2]. \quad [54]$$

Substituting in [54] from [50] and [51], we obtain

$$K_{in} = V\frac{\rho_1}{2}v_2^2\left[\frac{4\beta^2 + \beta - 5}{(2\beta + 1)^2} - \epsilon_2\right] \quad [55]$$

Adding [53] and [55], the net fluid kinetic energy is found to be

$$K = V\frac{\rho_1}{2}v_2^2\left(3\frac{\beta - 1}{2\beta + 1} - \epsilon_2\right), \quad [56]$$

which confirms [52].

We may also use Bernoulli's equation in averaged form, [37], with $C(t)$ equal to the pressure at infinity, p_∞, to determine the mean fluid pressure inside the sphere:

$$\frac{p_1}{\rho} = \frac{p_\infty}{\rho} + \frac{\partial\Phi}{\partial t} - \frac{v_1^2}{2} - \frac{(\epsilon_1\beta - 1)}{2}(v_1^2 - v_2^2). \quad [57]$$

Since the potential pattern moves with the sphere, we have, using [41] inside the sphere and substituting from [45]:

$$\frac{\partial\Phi}{\partial t} = -v_2 \cdot \nabla\Phi = -\frac{d}{3}v_2 = -v_2^2\frac{\beta - 1}{2\beta + 1}. \quad [58]$$

Substituting from [51] and [58] in [57] we obtain, after some algebra,

$$\frac{p_1}{\rho_1} = \frac{p_\infty}{\rho_1} - \frac{v_2^2}{2}\left[\frac{9\beta}{\epsilon_1(2\beta + 1)^2} - 1\right], \quad [59]$$

which reduces, for a Maxwellian dispersion, to

$$\frac{p_1}{\rho_1} = \frac{p_\infty}{\rho_1} - \frac{v_2^2}{4}\epsilon_2. \quad [60]$$

In all of the above it was assumed that the particles composing the sphere were small enough that the continuum description could be used right up to the surface of the sphere.

2. Flow past a stationary porous sphere

Imposing a velocity $-\mathbf{v}_2$ on the entire flow analyzed above, we obtain a flow past a stationary porous sphere. At infinity the fluid velocity is $-\mathbf{v}_2$ and the pressure p_∞.

In this case the average velocity of fluid in the pores of the sphere follows from [51] as

$$\mathbf{v}_1 = \frac{-3\mathbf{v}_2}{\epsilon_1(2\beta + 1)}. \quad [61]$$

For a Maxwellian dispersion this velocity is the same as the value at infinity, but some flow is diverted around the sphere because the particles occupy part of its volume.

The averaged Bernoulli equation, [37], for the fluid is now, between infinity and the interior of the sphere,

$$\frac{p_\infty}{\rho_1} + \frac{v_2^2}{2} = \frac{p_1}{\rho_1} + \epsilon_1\beta v_1^2. \quad [62]$$

Use of [62] in [61] confirms [59] and serves to illustrate the validity of the "extra" term that appears in [37] as a result of both the averaging procedures and the convection of the dipole pattern with the particles.

3. Motion of a porous cylinder normal to its axis in an infinite fluid

The analysis follows the procedure already used for the moving sphere, borrowing the appropriate standard results from classical field theory.

Equation [38] is still valid. The flow outside the cylinder is described in cylindrical coordinates by

$$\Phi = d\frac{a\cos\theta}{2R} \qquad [63]$$

and

$$v_R = \frac{a^2}{2R^2} d\cos\theta, \quad v_\theta = \frac{a^2}{2R^2} d\sin\theta. \qquad [64]$$

On the surface of the cylinder, $R = a$:

$$\Phi = \frac{d}{2} a\cos\theta = \frac{d}{2}x, \quad v_R = \frac{d}{2}\cos\theta, \quad v_\theta = \frac{d}{2}\sin\theta. \qquad [65]$$

From continuity of flux normal to the surface of the cylinder,

$$\mathbf{v}_2 = \frac{\mathbf{d}}{2}\frac{\beta+1}{\beta-1}. \qquad [66]$$

Equations [38], [65] and [66] confirm [25].

From [26], [38] and [66],

$$\mathbf{v}_1 = \mathbf{v}_2\left[1 - \frac{2}{\epsilon_1(\beta+1)}\right]; \qquad [67]$$

which reduces, for a Maxwellian dispersion, to

$$\mathbf{v}_1 = -\frac{\epsilon_2}{4-\epsilon_2}\mathbf{v}_2 \qquad [68]$$

[which differs from the result ($\mathbf{v}_1 = 0$) obtained for the moving sphere].

The added mass coefficient, based on the total volume of the cylinder, is

$$2\frac{\beta-1}{\beta+1} - \epsilon_2 \qquad [69]$$

which becomes, for a Maxwellian dispersion,

$$\frac{\epsilon_2}{2}\cdot\frac{1+\frac{\epsilon_2}{2}}{1-\frac{\epsilon_2}{4}} \qquad [70]$$

with, again, the right limits as $\epsilon_2 \to 0$ or $\epsilon_2 \to 1$.

Other examples

The three examples worked out above were inspired by well-known problems in classical field theory. Further solutions can be developed by adapting other known results from analogous problems in electromagnetic theory, heat transfer by conduction, ideal fluid mechanics etc.

EQUATION OF MOTION FOR THE FLUID

Since [37] is valid between any points in the macroscopic flow which are each at a uniform state, it is reasonable to hypothesize that it is a differentiable function whenever a "continuum" approach is appropriate (perhaps mathematicians can provide assessments of the degree of approximation involved). This is equivalent to regarding continuous changes as a series of small steps between uniform states.

Previously, Wallis (1990) derived [37] from Geurst's equation of motion for the fluid. Therefore, the latter can be deduced from the former merely by reversing the logical process. This will now be done, using a slightly different format that actually leads to a simpler result.

We take the gradient of [37], making use of [25] and standard expansions for the gradient of velocity squared:

$$\frac{\partial \mathbf{v}_1}{\partial t} + \frac{\partial}{\partial t}(E\mathbf{w}) + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \times \nabla \times \mathbf{v}_1 + E(\mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \times \nabla \times \mathbf{v}_1 - \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 - \mathbf{v}_2 \times \nabla \times \mathbf{v}_2) + \mathbf{w} \cdot \left(\mathbf{v}_1 - \frac{\mathbf{w}}{2}\right)\nabla E + \frac{\nabla p_1}{\rho_1} - \mathbf{g} = 0. \quad [71]$$

Taking the curl of [25] gives

$$\nabla \times \mathbf{v}_1 = -\nabla \times (E\mathbf{w}) = \mathbf{w} \times \nabla E - E\nabla \times \mathbf{w}. \quad [72]$$

When [72] is used and it is realized that

$$\mathbf{v}_1 \times (\mathbf{w} \times \nabla E) = (\mathbf{v}_1 \cdot \nabla E)\mathbf{w} - (\mathbf{v}_1 \cdot \mathbf{w})\nabla E, \quad [73]$$

[71] becomes

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + E\left(\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 - \frac{\partial \mathbf{v}_2}{\partial t} - \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + \mathbf{w} \times \nabla \times \mathbf{v}_2\right) + \mathbf{w}\left(\frac{\partial E}{\partial t} + \mathbf{v}_1 \cdot \nabla E\right) - \frac{1}{2} w^2 \nabla E + \frac{\nabla p_1}{\rho_1} - \mathbf{g} = 0. \quad [74]$$

Equation [74] resembles the usual equation of motion for the fluid but contains additional terms, involving the exertia, that describe inertial coupling between the phases.

The first term in parentheses is the relative acceleration together with a "lift" component that deserves further study. Such a term is often called an "added mass force".

The second term in parentheses is the rate of change of the exertia in a system moving at the average fluid velocity. By invoking the continuity equation for the fluid it may even be reduced to a single term:

$$\frac{\partial E}{\partial t} + \mathbf{v}_1 \cdot \nabla E = \frac{\partial E}{\partial \epsilon_1}\left(\frac{\partial \epsilon_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \epsilon_1\right) = -\epsilon_1 \frac{\partial E}{\partial \epsilon_1} \nabla \cdot \mathbf{v}_1. \quad [75]$$

The term involving w^2 may be related (Wallis 1989a, equation [3.16]) to the pressure difference between the phases (strictly speaking, we also need a proof that this is valid for unsteady motion):

$$-\frac{w^2 \nabla E}{2} = -\frac{w^2}{2}\frac{\partial E}{\partial \epsilon_1}\nabla \epsilon_1 = \frac{p_1 - p_2}{\rho_1 \epsilon_1}\nabla \epsilon_1. \quad [76]$$

A pressure difference term of this form has been postulated by many authors (e.g. Banerjee & Chan 1980).

Equation [74] is compatible with the equations presented in various forms by Geurst (1986), who showed that the terms involving exertia were objective and independent of the coordinate system.

EQUATION OF MOTION OF THE PARTICLES

We start with the momentum conservation equation for the combined phases, using the momentum density and combined momentum flux and stress tensor derived by Wallis (1989a, equation [3.152]):

$$\frac{\partial}{\partial t}(\epsilon_1 \rho_1 \mathbf{v}_1 + \epsilon_2 \rho_2 \mathbf{v}_2) + \nabla \cdot (\epsilon_1 \rho_1 \mathbf{v}_1 \mathbf{v}_1 + \epsilon_1 \rho_2 \mathbf{v}_2 \mathbf{v}_2 + \epsilon_1 \rho_1 E \mathbf{w}\mathbf{w}) + \nabla(\epsilon_1 p_1 + \epsilon_2 p_2) - (\epsilon_1 \rho_1 + \epsilon_2 \rho_2)\mathbf{g} = \epsilon_2 \mathbf{f}_2; \quad [77]$$

$\mathbf{f}_2$ is an external force acting on the particles per unit particle volume.

In addition, it will be useful to have the continuity equations for each phase:

$$\frac{\partial}{\partial t}(\epsilon_1 \rho_1) + \nabla \cdot (\epsilon_1 \rho_1 \mathbf{v}_1) = 0 \quad [78]$$

and

$$\frac{\partial}{\partial t}(\epsilon_2\rho_2) + \nabla\cdot(\epsilon_2\rho_2\mathbf{v}_2) = 0. \qquad [79]$$

Using [78] and [79] in the usual way in [77] and subtracting $\epsilon_1\rho_1$ times [74] from it, incorporating [76] as well, we get

$$\epsilon_2\rho_2\left(\frac{\partial\mathbf{v}_2}{\partial t} + \mathbf{v}_2\cdot\nabla\mathbf{v}_2\right) - \epsilon_1\rho_1 E\left(\frac{\partial\mathbf{v}_1}{\partial t} + \mathbf{v}_1\cdot\nabla\mathbf{v}_1 - \frac{\partial\mathbf{v}_2}{\partial t} - \mathbf{v}_2\cdot\nabla\mathbf{v}_2 + \mathbf{w}\times\nabla\times\mathbf{v}_2\right)$$
$$-\epsilon_1\rho_1\mathbf{w}\left(\frac{\partial E}{\partial t} + \mathbf{v}_1\cdot\nabla E\right) + \epsilon_2\nabla p_2 + \nabla\cdot(\epsilon_1\rho_1 E\mathbf{w}\mathbf{w}) = \epsilon_2(\mathbf{f}_2 + \rho_2\mathbf{g}). \qquad [80]$$

Note that there is no requirement that phase 2 be incompressible.

Equation [80] could be divided by ϵ_2 and called the "equation of motion of phase 2". The second and third terms, that have changed sign from [74], could be ascribed to a "mutual inertial coupling force".

The only remaining term requiring explanation is the final one on the l.h.s. It has the appearance of a stress tensor resulting from the relative motion and might at first sight appear to be of the "Reynolds stress" type that would be happier if located in [74]. A partial explanation for why it appears in [80] instead is that the "pressure" p_2 does not completely describe the state of stress of the particles which have their own tensor stress system, influenced by the relative motion (Wallis 1989a, p. 297). p_2 is actually the bulk stress in the particles but differs from the principal stresses which are oriented along and perpendicular to the relative motion and are intimately related to the Reynolds stresses in the fluid.

The terms involving E in [80] may be arranged as follows:

$$-\epsilon_1\rho_1\left(\frac{\partial}{\partial t}E\mathbf{w}\right) - \epsilon_1\rho_1 E(\mathbf{v}_1\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}_2 + \mathbf{w}\times\nabla\times\mathbf{v}_2) - \epsilon_1\rho_1\mathbf{w}\mathbf{w}_1\cdot\nabla E$$
$$+\nabla\cdot(\epsilon_1\rho_1 E\mathbf{v}_1\mathbf{w}) - \nabla\cdot(\epsilon_1\rho_1 E\mathbf{v}_2\mathbf{w}). \qquad [81]$$

The first divergence term may be expanded:

$$\nabla\cdot(\epsilon_1\rho_1 E\mathbf{v}_1\mathbf{w}) = E\mathbf{w}\nabla\cdot(\epsilon_1\rho_1\mathbf{v}_1) + \epsilon_1\rho_1\mathbf{v}_1\cdot\nabla(E\mathbf{w})$$
$$= -E\mathbf{w}\frac{\partial}{\partial t}(\epsilon_1\rho_1) + \epsilon_1\rho_1 E\mathbf{v}_1\cdot\nabla\mathbf{w} + \epsilon_1\rho_1\mathbf{w}\mathbf{v}_1\cdot\nabla E; \qquad [82]$$

and, moreover

$$\mathbf{w}\cdot\nabla\mathbf{v}_2 + \mathbf{w}\times\nabla\times\mathbf{v}_2 = \mathbf{w}\cdot(\nabla\mathbf{v}_2)^{\mathrm{T}}. \qquad [83]$$

Using [82] and [83] in [81], all the terms involving E in [80] can be expressed as

$$-\frac{\partial}{\partial t}(\epsilon_1\rho_1 E\mathbf{w}) - \epsilon_1\rho_1 E\mathbf{w}\cdot(\nabla\mathbf{v}_2)^{\mathrm{T}} - \nabla\cdot(\epsilon_1\rho_1 E\mathbf{v}_2\mathbf{w}),$$

which is the same as the "mutual force" derived by Geurst (1986) but does not have a direct physical interpretation in this form.

Clearly, care must be used with consistent definitions and interpretations. The "mutual force" that appears in [74] and [80] differs from Geurst's, although the overall equations are the same as his, because he includes the $\nabla\cdot(\epsilon_1\rho_1 E\mathbf{w}\mathbf{w})$ term in the equation for the fluid, rather than the one for the particles as we have done. The present formulation is simpler in appearance and may be more amenable to mechanistic interpretation.

Many further manipulations of these equations may be performed to demonstrate internal consistency and derive new results. For example, starting with the equations of motion it can be deduced (Geurst 1985; Wallis 1989b) that if $\nabla\times\mathbf{f}_2 = 0$, the two composite velocities, $\boldsymbol{\pi}_1 = \mathbf{v}_1 + E\mathbf{w}$ and $\boldsymbol{\pi}_2 = \mathbf{v}_2 - (\rho_1\epsilon_1/\rho_2\epsilon_2)E\mathbf{w}$ are each equal to the (negative) gradient of two macroscopic potentials, Φ and η, which appear directly from the variational approach used by Geurst.

It is useful to check the equations in the dilute limit, $\epsilon_2 \to 0$. We set

$$E = \epsilon_2 C, \qquad [84]$$

where C is the usual "added mass coefficient" and a constant. Neglecting terms of second order in ϵ_2, [84], [78] and [80] reduce to

$$\nabla \cdot \mathbf{v}_1 = 0, \qquad [85]$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + \frac{\nabla p_1}{\rho_1} - \mathbf{g} = 0 \qquad [86]$$

and

$$\rho_2\left(\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2\right) - \rho_1 C\left(\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1 \nabla \mathbf{v}_1 - \frac{\partial \mathbf{v}_2}{\partial t} - \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + \mathbf{w} \times \nabla \times \mathbf{v}_1\right)$$
$$- \rho_1 C \mathbf{w} \frac{1}{V_p}\left(\frac{\partial V_p}{\partial t} + \mathbf{v}_2 \cdot \nabla V_p\right) + \nabla p_1 = \mathbf{f}_2 + \rho_2 \mathbf{g}, \qquad [87]$$

where [79] was used and V_p is the volume of a typical particle with assumed constant shape. Similar results were derived by Geurst (1986) for spherical particles. The first term involving C in [87] has the same form as deduced by Drew & Lahey (1987) and the second appears in Lhuillier (1982) for the case of a bubble. Although a "lift force" appears from the mathematics, it is unlikely to be important for potential flows since $\nabla \times \mathbf{v}_1$ vanishes in the dilute limit $E \to 0$ by virtue of [25].

DISCUSSION

Up to [70], the arrays of particles analyzed had, at least locally, a fixed structure which determined β (or E). Bernoulli's equation, [30], being valid throughout an entire connected region, enabled relations such as [34], [35] or [37] to provide a bridge between properties in different parts of a flow with different void fractions and structure. The developments after [70] followed from the hypothesis that small changes between adjacent regions could occur in such a way that [37] was differentiable. In performing this differentiation it was implied that the exertia, E, depended only on the particle concentration, ϵ_2, as it does, for example, in a Maxwellian assembly of spheres. It was also assumed that the particles all have the same velocity in the local averaging volume in which macroscopic properties are defined.

A more sophisticated treatment would have to consider at least two additional features:

(a) Motion of particles relative to each other as a result of hydrodynamic interactions and possible collisions.
(b) Changes in the arrangement of the particles, especially in flow fields with large strains.

It might be fortunate that these effects are small because of "relaxation" mechanisms that drive the assembly towards an equilibrium state with small deviations in particle velocity and a "random" arrangement for which E is isotropic and depends only on ϵ_2. Otherwise there are prospects of the properties of the dispersion depending on its entire past history, which could transfer energy to random (or structured) particle velocity fluctuations and orient the particles in preferred geometrical patterns (e.g. rows or columns parallel to the axes of extensive strain). Such dispersions would be much more difficult to analyze and would not have "equations of motion" that could be expressed solely in terms of local conditions. An indication that these effects may be real is given by the observation (Geurst 1985; Prosperetti & Satrape 1990) that Geurst's equations are not well-posed and lead to instabilities if E is expressed as a function of ϵ_2 using classical results such as [8] or [46].

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